

A candidate proof of the Han–Xiong 1/2-conjecture

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Abstract

We present a candidate proof that the integer-exponent trace of the fractional Gaussian binomial coefficient is coefficientwise largest at $r = 1/2$, for every positive rational parameter r and every positive integer lower index k . An exact trace expansion reduces the comparison to $r = 1/(2m)$. Same-index pairing leaves explicit deficits, which are bounded by adjacent even-index summands. These pairs are disjoint for $m \geq 3$; a weighted identity handles their overlap for $m = 2$. A counting estimate covers the upper index boundary, and denominator comparisons reduce the remaining range to three rational kernels with denominator $(q; q)_4$. The classical restricted-partition formula gives a cubic approximation with an explicit linear error bound, yielding uniform coefficient bounds for all three kernels. The proof uses no finite parameter cutoff or symbolic case enumeration as a premise.

Preprint; not externally peer reviewed. AI use and verification are described in the appendix.

Introduction and proof strategy

Han and Xiong [1] conjectured that, among the integer-exponent traces of fractional Gaussian binomial coefficients with positive rational parameter, the trace at $r = 1/2$ is coefficientwise maximal. The proposed theorem below proves this comparison for every positive rational r and every positive lower index k . The numerator convention is $\prod_{i=1}^k (1 - q^{i+r})$; it is important that the product starts at $i = 1$. Section 1 defines the trace and states the theorem precisely.

The existing reductions make the unresolved part quite specific. Ono [2] reduced the problem to the canonical parameters $r = 1/(2m)$, proved structural comparison results, and verified the conjecture for all positive rational parameters when $k \leq 200$ by exact computation. Andrews and El Bachraoui [3] proved adjacent-pair estimates for $m \geq 2$ and $k \leq 6m + 2$, as well as $(m, k) = (2, 15)$; their result gives a non-computational proof for all parameters when $k \leq 15$. We use neither of these finite ranges as a premise. The reduction in [2] and the adjacent-pair strategy in [3] are prior work. The scope of [3] stated here is taken from its published abstract; its full text was unavailable for the present comparison.

Proof strategy. After the canonical reduction, we pair every summand of $T_{1/(2m),k}$ with the summand of $T_{1/2,k}$ having the same index. What remains is a sum of unused positive terms minus explicit deficits. We compensate a deficit at index $2m\ell$ using the two nearest unused even indices, $2m\ell - 2$ and $2m\ell + 2$. For $m \geq 3$ these pairs are disjoint. For $m = 2$ neighboring pairs overlap, so a weighted identity accounts for every positive term before the local estimates are summed.

The local estimates have two parts. Close to the upper index boundary, the right neighbor can be absent; a counting argument bounds the deficit using the left neighbor alone. Farther from that boundary, denominator comparisons reduce the local inequalities to three rational series with denominator $D_4 = (q; q)_4$. The classical exact formula for $[q^n]D_4^{-1}$, expressed as a cubic polynomial with an explicit linear error bound, then proves each kernel inequality uniformly.

All polynomial bounds and all finite initial coefficients needed for the proof are displayed in the manuscript.

Reading guide. Section 2 specifies exactly which local estimates suffice and explains their weights. Section 3 treats the boundary range. Section 4 handles the other range and lists the three kernel targets. Sections 5–7 prove those targets, using one common rule to generate the polynomial lower bounds. Section 8 completes the theorem. The appendix describes AI use, review status, and supplementary arithmetic checks.

1 Integer traces and reduction to primitive parameters

All generating functions in this paper are formal. For a nonnegative integer t , put

$$D_t(q) = (q; q)_t = \prod_{i=1}^t (1 - q^i), \quad D_0(q) = 1. \quad (1.1)$$

For a positive rational number r , expand the numerator below in a suitable fractional power of q . The operator $\text{Tr}_{\mathbb{Z}}$ retains precisely the terms whose exponents are integers. In particular, the numerator product starts at $i = 1$:

$$T_{r,k}(q) = \text{Tr}_{\mathbb{Z}} \left(\frac{\prod_{i=1}^k (1 - q^{i+r})}{D_k(q)} \right). \quad (1.2)$$

For ordinary formal power series F and G , write $F \succeq G$ if every coefficient of $F - G$ is nonnegative. Multiplication by a series with nonnegative coefficients preserves this order. Han and Xiong introduced the comparison in Theorem 1.1 as their 1/2-conjecture [1]. The reduction to primitive parameters was established by Ono [2]; we give its short proof here to make the argument self-contained. Adjacent-pair comparisons also occur in the work of Andrews and El Bachraoui [3].

Theorem 1.1. For every $r \in \mathbb{Q}_{>0}$, every integer $k \geq 1$, and every integer $n \geq 0$, one has

$$[q^n] \left(T_{1/2,k}(q) - T_{r,k}(q) \right) \geq 0. \quad (1.3)$$

1.1 The exact trace formula

The finite q -binomial theorem, followed by cancellation of D_k , gives

$$\frac{\prod_{i=1}^k (1 - q^{i+r})}{D_k} = \sum_{s=0}^k \frac{(-1)^s q^{s(s+1)/2+rs}}{D_s D_{k-s}}. \quad (1.4)$$

Suppose $r = u/b$ is in lowest terms, with u, b positive integers. The denominators on the right contain only integer powers of q . Since $s(s+1)/2$ is an integer, the s -th summand survives the trace precisely when b divides s . Consequently

$$T_{u/b,k}(q) = \sum_{0 \leq s \leq k, \, b|s} \frac{(-1)^s q^{s(s+1)/2+(u/b)s}}{D_s D_{k-s}}. \quad (1.5)$$

1.2 The even-core comparison

Put $\Lambda = \text{lcm}(b, 2)$, so Λ is even, and let $r^b = 1/\Lambda$. We claim

$$T_{1/\Lambda, k} \succeq T_{u/b, k}. \quad (1.6)$$

First note the elementary shift comparison: if $1 \leq s \leq k$, $d \geq 0$ is an integer, and $E \geq 0$ is an integer, then

$$\frac{q^E - q^{E+d}}{D_s D_{k-s}} \succeq 0. \quad (1.7)$$

Indeed, for $d > 0$ the numerator contains $1 - q^d$; cancel $1 - q$ against the factor of D_s and use $(1 - q^d)/(1 - q) = 1 + q + \dots + q^{d-1}$. For $d = 0$ the difference is zero.

If b is even, the surviving indices in (1.5) are exactly the multiples of $\Lambda = b$, and all their signs are positive. The same indices occur in $T_{1/\Lambda, k}$. Their exponents differ by the nonnegative integer $d = (u/b - 1/\Lambda)s$, so (1.7) compares each pair. The $s = 0$ terms are equal.

If b is odd, the positive summands of (1.5) have even indices divisible by b , hence indices divisible by $\Lambda = 2b$. These are exactly the indices of $T_{1/\Lambda, k}$, whose exponents again occur earlier or at the same degree. The remaining summands of $T_{u/b, k}$ have negative signs. Subtracting them adds nonnegative series. This proves (1.6), including the case $b = 1$.

It follows that Theorem 1.1 is equivalent to proving, for all positive integers m and k ,

$$T_{1/2, k} \succeq T_{1/(2m), k}. \quad (1.8)$$

The necessity is immediate by choosing $r = 1/(2m)$. Conversely, $\Lambda = 2m$ in (1.6), so (1.8) and transitivity give (1.3) for every positive rational r . The case $m = 1$ in (1.8) is equality. We henceforth take $m \geq 2$.

2 Global allocation of adjacent positive summands

Symbol	Role in the local comparison
k	Fixed lower index of the Gaussian binomial coefficient
m, ℓ	Canonical parameter $1/(2m)$ and deficit index
$s = 2m\ell$	Index of the matched term at the center of a deficit
$h = k - s$	Distance of that center from the upper index boundary
$a = m\ell$	Compressed parameter used when $m \geq 3$
n	Degree of the coefficient being estimated
x	Nonnegative translated parameter in the kernel proofs

Fix $k \geq 1$ throughout this section and the local estimates that follow. The notation A_s suppresses this fixed dependence on k ; thus $A_s = A_{k, s}$. For every even index s in $[0, k]$, define A_s by (2.1), and set it equal to zero for even indices outside this range. Put $J = \lfloor k/(2m) \rfloor$ and define $B_{m, \ell}$ for $0 \leq \ell \leq J$:

$$A_s = \frac{q^{s(s+2)/2}}{D_s D_{k-s}}, \quad B_{m, \ell} = \frac{q^{2m^2\ell^2 + (m+1)\ell}}{D_{2m\ell} D_{k-2m\ell}}. \quad (2.1)$$

Formula (1.5) gives

$$T_{1/2,k} = \sum_{0 \leq s \leq k, 2|s} A_s, \quad T_{1/(2m),k} = \sum_{\ell=0}^J B_{m,\ell}. \quad (2.2)$$

For $1 \leq \ell \leq J$, the matched positive term is $A_{2m\ell} = q^{(m-1)\ell} B_{m,\ell}$. Define its deficit by

$$C_{m,\ell} = (1 - q^{(m-1)\ell}) B_{m,\ell}, \quad \Delta_{m,k} = T_{1/2,k} - T_{1/(2m),k}. \quad (2.3)$$

$$B_{m,\ell} - A_{2m\ell} = C_{m,\ell}.$$

Thus $C_{m,\ell}$ is exactly the deficit left by this same-index pairing. It is a nonnegative series: cancel $1 - q$ from $D_{2m\ell}$ in $(1 - q^{(m-1)\ell}) B_{m,\ell}$, leaving a finite geometric sum and nonnegative reciprocal factors. The nearest even indices not used in the pairing are $2m\ell \pm 2$, which explains the choice of neighbors in the estimates below.

Let $U_{m,k}$ be the set of positive even indices at most k that are not divisible by $2m$. Cancelling the $s = 0$ terms and the matched terms gives the exact identity

$$\Delta_{m,k} = \sum_{s \in U_{m,k}} A_s - \sum_{\ell=1}^J C_{m,\ell}. \quad (2.4)$$

When $J = 0$, the second sum is empty and positivity follows at once. The arguments below address $J \geq 1$ and explicitly account for how often each positive summand is used.

For example, when $m = 2$, the first two deficits use the pairs (A_2, A_6) and (A_6, A_{10}) , so the term A_6 would be counted twice. When $m = 3$, the corresponding pairs are (A_4, A_8) and (A_{10}, A_{14}) , with no overlap. The following identities turn this distinction into an exact accounting argument; the examples are only an explanation of the design.

2.1 The shared-neighbor case $m = 2$

In this subsection let $J = \text{floor}(k/4)$, write $C_j = C_{2,j}$, and put $R_j = A_{4j-2}$ for every $j \geq 1$. Then

$$\Delta_{2,k} = \sum_{j=1}^{J+1} R_j - \sum_{j=1}^J C_j. \quad (2.5)$$

The last term is $R_{J+1} = A_{4J+2}$; it vanishes when $4J + 2 > k$. The local comparisons we shall establish are

$$2A_2 + A_6 \succeq 2C_1 \quad (k \geq 4), \quad (2.6)$$

$$A_{4j-2} + A_{4j+2} \succeq 2C_j \quad (2 \leq j \leq J). \quad (2.7)$$

To account for the overlap, multiply the target difference by two. Each positive term then has two copies available. An interior term is assigned once to each neighboring deficit; the extra copy at the left endpoint is included in the first comparison, and one copy at the right endpoint remains unused and nonnegative. This gives the exact identity below, including $J = 1$, when the middle sum is empty:

$$2\Delta_{2,k} = (2R_1 + R_2 - 2C_1) + \sum_{j=2}^J (R_j + R_{j+1} - 2C_j) + R_{J+1}. \quad (2.8)$$

For $J = 2$, for instance, the same identity reads

$$2\Delta_{2,k} = (2R_1 + R_2 - 2C_1) + (R_2 + R_3 - 2C_2) + R_3.$$

The first bracket explains the asymmetric weight in (2.6). Each bracket is nonnegative by the stated local comparisons, and the last term is nonnegative. Terms with indices greater than k are zero by convention. This ensures that no positive summand is spent more often than allowed.

2.2 Disjoint neighbors for $m \geq 3$

For $m \geq 3$, it suffices instead to establish the unweighted comparisons

$$A_{2m\ell-2} + A_{2m\ell+2} \succeq C_{m,\ell} \quad (1 \leq \ell \leq J). \quad (2.9)$$

Both neighbors, whenever their indices are in $[0,k]$, belong to $U_{m,k}$. Different deficits do not share a neighbor. Indeed, an equality between a right neighbor and another left neighbor would require $2m(\ell' - \ell) = 4$, impossible for $m \geq 3$; equality between neighbors of the same kind forces $\ell' = \ell$. Thus every valid positive summand in (2.9) is used at most once. Summing (2.9), and adding the unused A terms from (2.4), proves $\Delta_{m,k} \succeq 0$. The convention $A_s = 0$ outside $[0,k]$ includes the upper boundary automatically.

3 Counting estimates near the upper index boundary

The variable $h = k - s$ measures how much room remains above the deficit center. When $h = 0$ or $h = 1$, the right neighbor is absent, so we first seek estimates using only the left neighbor. For larger h , both neighbors can be put over a useful common denominator. The following table records the ranges that will be proved in Sections 3 and 4; it also shows in advance why they suffice.

Case	Counting (Section 3)	Kernel (Section 4)
$m \geq 3, a = m\ell$	$0 \leq h \leq 2a + 1$	$h \geq 2a + 2$
$m = 2, j \geq 2$	$0 \leq h \leq 6j - 1$	$h \geq 4j + 2$
$m = 2, j = 1$	$0 \leq h \leq 5$	$h \geq 6$

The first and third rows are consecutive ranges. The second row has an overlap: $4j + 2 \leq 6j$ for $j \geq 2$, so it too leaves no integer $h \geq 0$ uncovered. A kernel inequality below always means coefficientwise nonnegativity after division by D_4 , not nonnegativity of every coefficient of the numerator polynomial.

For a positive integer t , write $[t]_q = 1 + q + \cdots + q^{t-1}$, and set $[0]_q = 0$. We prove two local estimates by cancelling D_2^2 from a common denominator. Each remaining reciprocal factor is a geometric series with nonnegative coefficients.

3.1 The estimate needed when $m = 2$

Lemma 3.1. Let $j \geq 1$, $s = 4j$, and $h = k - s \geq 0$. If $h \leq 6j - 1$, then $A_{s-2} \succeq 2C_j$.

Proof. Bringing the two series to a common denominator gives

$$A_{s-2} - 2C_j = \frac{q^{8j^2-4j}N}{D_{4j}D_{h+2}}, \quad (3.1)$$

$$N = (1 - q^{4j-1})(1 - q^{4j}) - 2q^{7j}(1 - q^j)(1 - q^{h+1})(1 - q^{h+2}). \quad (3.2)$$

Among the consecutive integers $h + 1$ and $h + 2$, write the even one as $2v$ and the odd one as w . The hypothesis gives $v \leq 3j$. Exact cancellation yields

$$\frac{N}{D_2^2} = \frac{1}{1 - q^2} \left(\frac{[2j]_q [4j - 1]_q}{1 - q} - 2q^{7j} [j]_q [v]_{q^2} [w]_q \right). \quad (3.3)$$

Each coefficient of $[2j]_q [v]_{q^2} [w]_q$ is at most $2jv \leq 6j^2$: after choosing an exponent from each of the first two factors, there is at most one choice from the third factor for a prescribed total degree. Thus the negative term inside the parentheses has coefficients of magnitude at most $6j^2$ and first appears at degree $7j$.

To obtain a lower bound for the positive part, write its coefficient as an explicit count:

$$[q^n] \frac{[2j]_q [4j - 1]_q}{1 - q} = \#\{(u, t) : 0 \leq u \leq 2j - 1, 0 \leq t \leq 4j - 2, 2u + t \leq n\}.$$

For every $n \geq 7j$, this set contains the rectangle $0 \leq u \leq \lfloor 3j/2 \rfloor$, $0 \leq t \leq 4j - 2$. Indeed, its first coordinate is within range even when $j = 1$, and its largest possible $2u + t$ is at most $7j - 2$. Thus the whole rectangle has already entered the count before the negative term first appears, and it remains counted at every later degree. Its size is at least the upper bound $6j^2$ for a coefficient of the negative polynomial, because

$$(\lfloor 3j/2 \rfloor + 1)(4j - 1) \geq 6j^2. \quad (3.4)$$

Here are the two arithmetic checks in (3.4). If $j = 2b$ with $b \geq 1$, the difference of the two sides is $(3b + 1)(8b - 1) - 24b^2 = 5b - 1 \geq 0$. If $j = 2b + 1$ with $b \geq 0$, it is $(3b + 2)(8b + 3) - 6(2b + 1)^2 = b \geq 0$. Before degree $7j$ there is no negative term; from degree $7j$ onward the positive count dominates every negative coefficient. This proves $N/D_2^2 \succeq 0$. Both D_{4j} and D_{h+2} contain D_2 , so $D_2^2/(D_{4j}D_{h+2})$ is a product of reciprocal geometric factors and has nonnegative coefficients. Multiplying it by N/D_2^2 and by the monomial in (3.1) proves the assertion, including $h = 0$. $\square$

3.2 A uniform estimate for $m \geq 3$

Lemma 3.2. Let $m \geq 3$, $\ell \geq 1$, $a = m\ell$, and $h = k - 2a \geq 0$. If $h \leq 2a + 1$, then $A_{2a-2} \succeq C_{m,\ell}$.

Proof. Here $a \geq 3$ and $a - \ell \geq 1$. The common-denominator identity is

$$A_{2a-2} - C_{m,\ell} = \frac{q^{2a^2-2a}N}{D_{2a}D_{h+2}}, \quad (3.5)$$

$$N = (1 - q^{2a-1})(1 - q^{2a}) - q^{3a+\ell}(1 - q^{a-\ell})(1 - q^{h+1})(1 - q^{h+2}). \quad (3.6)$$

Again write the even member of $\{h+1, h+2\}$ as $2v$ and the odd member as w . Now $v \leq a+1$. Cancellation gives

$$\frac{N}{D_2^2} = \frac{1}{1-q^2} \left(\frac{[a]_{q^2}[2a-1]_q}{1-q} - q^{3a+\ell} [a-\ell]_q [v]_{q^2} [w]_q \right). \quad (3.7)$$

The polynomial multiplying the negative monomial has every coefficient at most $(a-\ell)v$. Since $\ell \geq 1$ and $v \leq a+1$, this is at most $(a-1)(a+1) = a^2 - 1$. No negative contribution occurs before degree $3a + \ell$.

For $n \geq 3a + \ell$, the positive term counts pairs (u, t) with $0 \leq u \leq a-1$, $0 \leq t \leq 2a-2$, and $2u + t \leq n$. Its admissible pairs include the rectangle $0 \leq u \leq \text{floor}(a/2)$, $0 \leq t \leq 2a-2$, because $\text{floor}(a/2) \leq a-1$ and $2u + t \leq 3a-2$. Moreover,

$$(\lfloor a/2 \rfloor + 1)(2a-1) \geq a^2 - 1. \quad (3.8)$$

For $a = 2b$ the difference between the two sides is $3b$; for $a = 2b+1$ it is $b+1$. Thus $N/D_2^2 \succeq 0$ at every degree. Multiplication by the nonnegative series $D_2^2/(D_{2a}D_{h+2})$ and the monomial factor in (3.5) proves the lemma. In particular, the proof includes $h=0$ and $h=1$, where the upper neighbor in (2.9) is absent. $\square$

4 Denominator comparisons and three rational kernels

4.1 A common denominator lemma

For nonnegative integers u, v , put $\Phi_{u,v} = 1/(D_u D_v)$. The following comparisons concern the complete rational series, rather than bare ratios of polynomials.

Lemma 4.1. If $s \geq 2$ and $h \geq s+2$, then

$$\Phi_{s-2, h+2} \succeq (1-q^{s-1})(1-q^s)\Phi_{s,h}, \quad \Phi_{s+2, h-2} \succeq \Phi_{s,h}. \quad (4.1)$$

Proof. The difference in the first comparison is

$$\frac{1}{D_{s-2}} \left(\frac{1}{D_{h+2}} - \frac{1}{D_h} \right) \succeq 0, \quad (4.2)$$

because adding the two geometric-series factors increases every coefficient. For the second comparison, if $h = s+2$ the denominator factors are simply interchanged, giving equality. If $h \geq s+3$, insert the intermediate term $\Phi_{s+1, h-1}$. The successive differences are

$$\Phi_{s+1, h-1} - \Phi_{s,h} = \frac{q^{s+1}(1-q^{h-s-1})}{D_{s+1}D_h}, \quad (4.3)$$

$$\Phi_{s+2, h-2} - \Phi_{s+1, h-1} = \frac{q^{s+2}(1-q^{h-s-3})}{D_{s+2}D_{h-1}}. \quad (4.4)$$

A numerator factor with exponent zero makes its difference zero. Otherwise, $1 \leq h-s-1 \leq h$ in (4.3), and $1 \leq h-s-3 \leq h-1$ in (4.4). The numerator factor therefore cancels one factor of D_h or D_{h-1} , leaving a monomial times nonnegative geometric series. This proves both differences nonnegative and establishes the lemma. $\square$

4.2 The variable kernel for $m = 2$

Take $j \geq 2$, $s = 4j$, $h = k - s$, and $e = 8j^2 - 4j$. When $h \geq s + 2$, all relevant indices are valid, and the exponent shifts are

$$A_{s-2} = q^e \Phi_{s-2, h+2}, \quad B_{2,j} = q^{e+7j} \Phi_{s,h}, \quad A_{s+2} = q^{e+16j+4} \Phi_{s+2, h-2}. \quad (4.5)$$

Since $C_j = (1 - q^j)B_{2,j}$, Lemma 4.1 gives

$$A_{s-2} + A_{s+2} - 2C_j \succeq q^e \Phi_{s,h} K_j(q), \quad (4.6)$$

$$K_j(q) = (1 - q^{4j-1})(1 - q^{4j}) - 2q^{7j} + 2q^{8j} + q^{16j+4}. \quad (4.7)$$

The order-preserving multiplication needed at this step is explicitly

$$q^e \Phi_{s,h} K_j = q^e \frac{D_4}{D_s D_h} \frac{K_j}{D_4}.$$

The first quotient has nonnegative coefficients because D_4 cancels inside D_s . The same argument will be used for the other two kernels. The choice of D_4 is useful for both algebraic and arithmetic reasons: it cancels from the denominators here, and the coefficients of $1/D_4$ admit a cubic approximation with an explicit linear error bound.

Thus $K_j/D_4 \succeq 0$ implies (2.7) in this range; this kernel inequality is proved in Lemma 6.1, Section 6. There is no missing h -range: Lemma 3.1 covers $h \leq 6j - 1$, and every remaining integer h satisfies $h \geq 6j \geq 4j + 2$.

4.3 The first-index kernel when $m = 2$

For $j = 1$, use the first-index comparison (2.6). If $h = k - 4 \leq 5$, Lemma 3.1 already gives $A_2 \succeq 2C_1$, which implies (2.6) after adding the nonnegative series $A_2 + A_6$. If $h \geq 6$, Lemma 4.1 instead gives

$$2A_2 + A_6 - 2C_1 \succeq q^4 \Phi_{4,h} K_*(q), \quad (4.8)$$

$$K_*(q) = 2(1 - q^3)(1 - q^4) - 2q^7(1 - q) + q^{20} = 2 - 2q^3 - 2q^4 + 2q^8 + q^{20}. \quad (4.9)$$

Consequently $K_*/D_4 \succeq 0$ proves (2.6) for $h \geq 6$. The fixed-kernel proof is Lemma 7.1, Section 7. Together with Lemma 3.1, these two h -ranges include every $k \geq 4$.

4.4 A single uniform kernel for all $m \geq 3$

Now let $m \geq 3$, $\ell \geq 1$, $a = m\ell$, $s = 2a$, $h = k - 2a$, and $e = 2a^2 - 2a$. In the range $h \geq 2a + 2$, the exact shifts are

$$A_{s-2} = q^e \Phi_{s-2, h+2}, \quad B_{m,\ell} = q^{e+3a+\ell} \Phi_{s,h}, \quad (4.10)$$

$$A_{s+2} = q^{e+8a+4} \Phi_{s+2, h-2}, \quad C_{m,\ell} = (1 - q^{a-\ell})B_{m,\ell}. \quad (4.11)$$

Applying Lemma 4.1 to the two positive terms yields

$$A_{s-2} + A_{s+2} - C_{m,\ell} \succeq q^e \Phi_{s,h} K_{a,\ell}(q), \quad (4.12)$$

$$K_{a,\ell}(q) = (1 - q^{2a-1})(1 - q^{2a}) - q^{3a+\ell} + q^{4a} + q^{8a+4}. \quad (4.13)$$

Define the kernel independent of ℓ by

$$Q_a(q) = (1 - q^{2a-1})(1 - q^{2a}) - q^{3a+1} + q^{4a} + q^{8a+4}. \quad (4.14)$$

Since $\ell \geq 1$, these two kernels satisfy

$$\frac{K_{a,\ell} - Q_a}{D_4} = \frac{q^{3a+1}(1 - q^{\ell-1})}{D_4} \succeq 0. \quad (4.15)$$

For $\ell = 1$ the difference is zero. For $\ell \geq 2$, cancel $1 - q$ and use the nonnegative polynomial $[\ell - 1]_q$. Thus $Q_a/D_4 \succeq 0$ for all $a \geq 3$ suffices for (2.9) in the range $h \geq 2a + 2$: first (4.15) gives $K_{a,\ell}/D_4 \succeq 0$, then multiply by the nonnegative series $q^e D_4/(D_s D_h)$ in (4.12). Lemma 3.2 covers the complementary range $h \leq 2a + 1$. The proof of the uniform kernel inequality, together with the other two required kernel inequalities, is supplied in Sections 5–7; Section 8 combines them with the allocations of Section 2 and the even-core comparison of Section 1.

4.5 The three remaining targets

The local comparisons have now been reduced to the following three statements. This table specifies which part of the global allocation each statement completes.

Kernel statement	Local comparison completed	Proof
$Q_a/D_4 \succeq 0$ ($a \geq 3$)	(2.9), for $m \geq 3$	Section 5
$K_j/D_4 \succeq 0$ ($j \geq 2$)	(2.7), for $m = 2$, $j \geq 2$	Section 6
$K_*/D_4 \succeq 0$	(2.6), for $m = 2$, $j = 1$	Section 7

Combined with the coverage table at the start of Section 3, these are the only remaining inequalities needed for the theorem. In particular, the next three sections estimate coefficients of fixed-denominator rational series; they no longer involve the original lower index k .

5 An elementary proof of the uniform kernel

5.1 An exact coefficient formula and its error bound

Let $p_4(t)$ be the coefficient of q^t in $1/D_4$ for integers t at least zero, and set $p_4(t) = 0$ for negative integers. Define the following polynomials on the real line:

$$\Pi(t) = \frac{2t^3 + 30t^2 + 135t + 175}{288}, \quad E(t) = \frac{t+5}{32} + \frac{17}{72}. \quad (5.1)$$

We use the classical exact restricted-partition formula below; an equivalent wave decomposition for p_4 is given by Sills and Zeilberger [4]. Partitions into parts at most four and into at most four parts have the same counting function by conjugation. For every integer $t \geq 0$,

$$p_4(t) = \Pi(t) + \frac{(-1)^t(t+5)}{32} + \frac{\delta_3(t)}{9} + \frac{\delta_4(t)}{8}, \quad (5.2)$$

where δ_3 has the successive values (1,0,-1) modulo 3 and δ_4 has the successive values (1,0,-1,0) modulo 4. Consequently $|p_4(t) - \Pi(t)| \leq E(t)$. To verify (5.2), one can multiply the following rational-function identity by its common denominator:

$$\begin{aligned} \frac{1}{D_4} &= \frac{1}{24(1-q)^4} + \frac{1}{8(1-q)^3} + \frac{59}{288(1-q)^2} + \frac{17}{72(1-q)} \\ &+ \frac{1}{32(1+q)^2} + \frac{1}{8(1+q)} + \frac{1+q}{9(1+q+q^2)} + \frac{1}{8(1+q^2)}. \end{aligned} \quad (5.3)$$

Coefficient extraction gives (5.2) for every integer $t \geq 0$, not just asymptotically. The last two periodic terms have absolute values at most $1/9$ and $1/8$, whose sum is $17/72$. The parity term has absolute value $(t+5)/32$; together these bounds give $|p_4(t) - \Pi(t)| \leq E(t)$. Thus the total error is bounded by an explicit linear function of t , rather than by a constant.

We will repeatedly use the following elementary consequence. At degree n , call an exponent-weight pair (e_i, w_i) active if $n \geq e_i$. Summing over the active pairs gives

$$\sum_i w_i p_4(n - e_i) \geq \sum_i (w_i \Pi(n - e_i) - |w_i| E(n - e_i)). \quad (5.4)$$

The bound applies to positive and negative weights alike: if $p_4(z) = \Pi(z) + \varepsilon(z)$ and $|\varepsilon(z)| \leq E(z)$, then $w\varepsilon(z) \geq -|w|E(z)$. Only active terms, with $z = n - e_i \geq 0$, enter this estimate; an inactive term contributes $p_4(n - e_i) = 0$ and is omitted. To make every later expansion reproducible on paper, define

$$f(z) = 2z^3 + 30z^2 + 135z + 175, \quad g(z) = 9z + 113.$$

For an ordered list of exponent-weight pairs and an initial segment of ν active pairs, define the lower bound by the single rule

$$288\mathcal{B}_\nu(n) = \sum_{i=1}^{\nu} (w_i f(n - e_i) - |w_i| g(n - e_i)).$$

Thus $f = 288\Pi$ and $g = 288E$. We call this bound $L_\nu(a, n)$ for the uniform kernel and $M_\nu(j, n)$ for the $m = 2$ kernel. The subscript ν counts active pairs; it is independent of the deficit index j . Every expanded polynomial and endpoint value below is obtained by this same rule.

5.2 The uniform kernel and its active ranges

Lemma 5.1. For every integer $a \geq 3$, the following formal series has nonnegative coefficients:

$$\frac{Q_a(q)}{D_4(q)}, \quad Q_a = (1 - q^{2a-1})(1 - q^{2a}) - q^{3a+1} + q^{4a} + q^{8a+4}. \quad (5.5)$$

Proof. In increasing order, the seven exponent-weight pairs are

$$(0, 1), (2a - 1, -1), (2a, -1), (3a + 1, -1), (4a - 1, 1), (4a, 1), (8a + 4, 1).$$

If exactly the first ν pairs are active, let $L_\nu(a, n)$ be the lower bound just defined for those pairs. By (5.4), $[q^n](Q_a/D_4) \geq L_\nu(a, n)$. Throughout the endpoint calculations put $x = a - 3 \geq 0$. The complete degree split is as follows:

Degrees	Bound used	Reason for nonnegativity
$0 \leq n < 3a + 1$	Exact product	Nonnegative reciprocal product
$3a + 1 \leq n \leq 4a - 2$	L_4	Positive derivative; check left endpoint
$n = 4a - 1$	L_5	Explicit positive parameter polynomial
$4a \leq n \leq 8a + 3$	L_6	Concave quadratic; check both endpoints
$n \geq 8a + 4$	L_7	Uniform cubic tail estimate

For $n < 3a + 1$, the extra monomials of Q_a have not contributed, and positivity is immediate from the exact product decomposition

$$\frac{(1 - q^{2a-1})(1 - q^{2a})}{D_4} = \frac{[2a - 1]_q [a]_{q^2}}{(1 - q^3)(1 - q^4)} \succeq 0. \quad (5.6)$$

5.3 The first three active ranges

For $3a + 1 \leq n \leq 4a - 2$, exactly four pairs are active. Before expanding, their common lower-bound rule reads

$$\begin{aligned} 288L_4(a, n) &= f(n) - g(n) \\ &\quad - f(n - 2a + 1) - g(n - 2a + 1) \\ &\quad - f(n - 2a) - g(n - 2a) \\ &\quad - f(n - 3a - 1) - g(n - 3a - 1). \end{aligned}$$

In particular, each negative weight also has a negative error term. Expanding this displayed cubic expression gives

$$\begin{aligned} 288L_4 &= -4n^3 + (42a - 60)n^2 + (-102a^2 + 408a - 318)n \\ &\quad + 86a^3 - 480a^2 + 978a - 862. \end{aligned} \quad (5.7)$$

As a real polynomial in n , its derivative satisfies

$$144 \frac{\partial L_4}{\partial n} = H_a(n) := -6n^2 + (42a - 60)n - 51a^2 + 204a - 159.$$

The quadratic H_a is concave, so it is bounded below on this interval by the lesser endpoint value. Those endpoint values, in terms of $x = a - 3$, are

$$H_a(3a + 1) = 54 + 156x + 21x^2, \quad H_a(4a - 2) = 54 + 102x + 21x^2.$$

The derivative is therefore positive on the whole interval: a concave quadratic lies above the line segment joining its endpoint values, and both values are positive. Consequently L_4 is increasing. Its minimum is at the left endpoint, where

$$288L_4(a, 3a + 1) = 2554 + 2904x + 696x^2 + 50x^3 > 0. \quad (5.8)$$

At $n = 4a - 1$ there are five active terms. Direct substitution gives

$$288L_5(a, 4a - 1) = 2666 + 3210x + 996x^2 + 94x^3 > 0. \quad (5.9)$$

For $4a \leq n \leq 8a + 3$, six terms are active, and the lower bound is

$$288L_6 = -6(a - 1)n^2 + (90a^2 - 120a)n - 170a^3 + 576a^2 - 294a - 580. \quad (5.10)$$

Here it is L_6 itself, rather than its derivative, that is concave in n : its second derivative is $-(a - 1)/24 < 0$. A concave function is at least the lesser of its two endpoint values on a closed interval. Those values are

$$\begin{aligned} 288L_6(a, 4a) &= 2804 + 3396x + 1038x^2 + 94x^3, \\ 288L_6(a, 8a + 3) &= 2534 + 3954x + 1476x^2 + 166x^3. \end{aligned} \quad (5.11)$$

Concavity therefore proves positivity throughout this closed interval.

5.4 The complete tail

For $n \geq 8a + 4$, all seven pairs are active. The translation $t = n - 9a + 2$ is chosen to remove the quadratic term in the cubic lower bound. Its range is $t \geq 6 - a \geq -a$; in particular, t need not be nonnegative. Expansion gives

$$144L_7 = t^3 - (6a^2 + 21)t + 78a^3 + 6a^2 - 156a - 301. \quad (5.12)$$

If $-a \leq t \leq 0$, then $t^2 \leq a^2$, so $t^2 - 6a^2 - 21 < 0$. Multiplication by $t \leq 0$ gives $t(t^2 - 6a^2 - 21) \geq 0$. For $t \geq 0$, the condition $a \geq 3$ gives $6a^2 + 21 \leq 25a^2/3$, and

$$t^3 - \frac{25}{3}a^2t + \frac{250}{27}a^3 = \left(t - \frac{5a}{3}\right)^2 \left(t + \frac{10a}{3}\right) \geq 0.$$

For $t \geq 0$ the factorization implies a lower bound $-250a^3/27 \geq -10a^3$ for $t^3 - (6a^2 + 21)t$. For negative t the lower bound zero just proved is stronger. Thus the single bound $t^3 - (6a^2 + 21)t \geq -10a^3$ holds on the entire tail, and (5.12) yields

$$144L_7 \geq 68a^3 + 6a^2 - 156a - 301 = 1121 + 1716x + 618x^2 + 68x^3 > 0. \quad (5.13)$$

The five degree ranges just treated are consecutive and cover every integer $n \geq 0$, including the single-point first active interval when $a = 3$. This proves Lemma 5.1. In fact the coefficients are strictly positive from $n = 3a + 1$ onward. $\square$

6 An elementary proof of the $m = 2$ variable kernel

Lemma 6.1. For every integer $j \geq 2$, $K_j/D_4 \succeq 0$, where

$$K_j = (1 - q^{4j-1})(1 - q^{4j}) - 2q^{7j} + 2q^{8j} + q^{16j+4}. \quad (6.1)$$

Proof. The ordered exponent-weight pairs are

$$(0, 1), (4j - 1, -1), (4j, -1), (7j, -2), (8j - 1, 1), (8j, 2), (16j + 4, 1).$$

Use the same lower-bound rule as in Section 5, retaining the absolute values of the weights in the error term. Write $M_\nu(j, n)$ for the bound with the first ν pairs active, and put $x = j - 2 \geq 0$. The degree ranges and the corresponding arguments are the following:

Degrees	Bound used	Reason for nonnegativity
$0 \leq n < 7j$	Exact product	Nonnegative reciprocal product
$7j \leq n \leq 8j - 2$	M_4	Positive derivative; check left endpoint
$n = 8j - 1$	M_5	Explicit positive parameter polynomial
$8j \leq n \leq 16j + 3$	M_6	Concave quadratic; check both endpoints
$n \geq 16j + 4$	M_7	Translated cubic and square factorization

For $n < 7j$ the extra monomials have not contributed, and nonnegativity follows from the exact product decomposition below.

$$\frac{(1 - q^{4j-1})(1 - q^{4j})}{D_4} = \frac{[4j - 1]_q [j]_{q^4}}{(1 - q^2)(1 - q^3)} \succeq 0. \quad (6.2)$$

For $7j \leq n \leq 8j - 2$, exactly four pairs are active. The weight -2 in the fourth pair must also be retained in the error term; explicitly,

$$\begin{aligned} 288M_4(j, n) &= f(n) - g(n) \\ &\quad - f(n - 4j + 1) - g(n - 4j + 1) \\ &\quad - f(n - 4j) - g(n - 4j) \\ &\quad - 2f(n - 7j) - 2g(n - 7j). \end{aligned}$$

Expansion gives

$$\begin{aligned} 288M_4 &= -6n^3 + (132j - 96)n^2 + (-780j^2 + 1368j - 516)n \\ &\quad + 1628j^3 - 3996j^2 + 3432j - 1266. \end{aligned} \quad (6.3)$$

The derivative of $288M_4$ is a concave quadratic in n . Its endpoint values are

$$\begin{aligned} 288 \partial_n M_4(j, 7j) &= 276 + 768x + 186x^2, \\ 288 \partial_n M_4(j, 8j - 2) &= 276 + 600x + 180x^2. \end{aligned}$$

As in Section 5.3, concavity of this derivative and positivity at both endpoints imply that the derivative is positive throughout the interval. Thus M_4 is increasing, with positive left endpoint

$$288M_4(j, 7j) = 6502 + 10260x + 4344x^2 + 578x^3 > 0. \quad (6.4)$$

At $n = 8j - 1$, exactly five terms are active, and

$$288M_5(j, 8j - 1) = 6750 + 11124x + 5208x^2 + 764x^3 > 0. \quad (6.5)$$

For $8j \leq n \leq 16j + 3$ there are six active terms. The lower bound is

$$288M_6 = -12jn^2 + (372j^2 - 168j - 72)n - 1444j^3 + 2148j^2 - 120j - 922. \quad (6.6)$$

Its second derivative is $-j/12$, so it is concave. Its endpoint values are

$$\begin{aligned} 288M_6(j, 8j) &= 7014 + 11688j + 5388j^2 + 764j^3, \\ 288M_6(j, 16j + 3) &= 4278 + 13044j + 8040j^2 + 1436j^3. \end{aligned} \quad (6.7)$$

Both are positive, proving the middle range.

For the entire tail $n \geq 16j + 4$, translate n to remove the quadratic term, and abbreviate the other coefficients by

$$\begin{aligned} t &= n - 18j + 1, & B &= 6j^2 + 24j + 16, & C &= 1380j^3 - 816j^2 - 1308j - 918. \\ 288M_7 &= 2t^3 - 6Bt + C. \end{aligned} \quad (6.8)$$

The tail begins at $t \geq 5 - 2j$. Although t may be negative, $\sqrt{B} \geq j$ implies $t + 2\sqrt{B} \geq 5 > 0$. The square factor in the following identity is therefore multiplied by a positive quantity throughout the tail:

$$2t^3 - 6Bt = 2(t - \sqrt{B})^2(t + 2\sqrt{B}) - 4B^{3/2}$$

It follows that $288M_7 \geq C - 4B\sqrt{B}$. To eliminate the square root, use $\sqrt{B} \leq 4j + 2$ for $j \geq 2$. Indeed, with $x = j - 2 \geq 0$, we have $(4j + 2)^2 - B = 10j^2 - 8j - 12 = 12 + 32x + 10x^2 > 0$. Consequently

$$\begin{aligned} 288M_7 &\geq C - 4B(4j + 2) = 1284j^3 - 1248j^2 - 1756j - 1046 \\ &= 722 + 8660x + 6456x^2 + 1284x^3 > 0. \end{aligned} \quad (6.9)$$

The five degree ranges cover every nonnegative integer, so Lemma 6.1 follows. The proof uses the displayed coefficient formula and polynomial inequalities, with no symbolic case enumeration as a premise. $\square$

7 The first-index kernel

Lemma 7.1. The series K_*/D_4 has nonnegative coefficients, where $K_* = 2 - 2q^3 - 2q^4 + 2q^8 + q^{20}$.

Proof. With $p_4(t) = 0$ for negative integers, the coefficient $c_n = [q^n](K_*/D_4)$ is exactly

$$c_n = 2p_4(n) - 2p_4(n - 3) - 2p_4(n - 4) + 2p_4(n - 8) + p_4(n - 20).$$

Substituting the exact formula (5.2) gives the following indexed initial values. This is a finite calculation with the same formula used for the tail; it requires no unstated coefficient computation.

n	0	1	2	3	4	5	6	7	8	9
c_n	2	2	4	4	6	6	8	6	10	8
n	10	11	12	13	14	15	16	17	18	19
c_n	10	8	12	8	12	8	12	8	12	6

(7.1)

For $n \geq 20$, all five pairs are active. Apply (5.4) to $(0, 2), (3, -2), (4, -2), (8, 2), (20, 1)$. Denote the lower bound by $F_*(n)$, so $c_n \geq F_*(n)$. The common rule gives

$$144F_*(n) = n^3 - 51n^2 + 801n - 3353. \quad (7.2)$$

Set $x = n - 20 \geq 0$. The same polynomial becomes

$$144F_*(n) = x^3 + 9x^2 - 39x + 267 = x^3 + \left(3x - \frac{13}{2}\right)^2 + \frac{899}{4} > 0. \quad (7.3)$$

This proves the complete tail and hence the lemma. The finite list (7.1) is fully displayed; no undisclosed finite verification is needed. $\square$

8 Completion of the theorem

For $m \geq 3$, Section 3 proves the local comparison when $h \leq 2a + 1$; Section 4 and Lemma 5.1 prove it when $h \geq 2a + 2$. These ranges are consecutive. The nonoverlapping allocation in Section 2 then yields the canonical comparison for every $m \geq 3$ and every k . When $k < 2m$ there is no missing primitive term, and the same global identity gives the comparison directly.

For $m = 2$ and $j \geq 2$, Section 3 covers $h \leq 6j - 1$, while Section 4 and Lemma 6.1 cover $h \geq 4j + 2$. There is no omitted integer between these ranges, since $4j + 2 \leq 6j$. For $j = 1$, the counting estimate covers $h \leq 5$ and Lemma 7.1 covers $h \geq 6$. The weighted summation identity in Section 2 therefore proves the comparison for all $k \geq 4$. For $k = 1, 2, 3$ there are no negative gap terms, so the result is immediate.

The case $m = 1$ is equality. Thus every canonical parameter $1/(2m)$ is covered. The parity reduction in Section 1 implies the full statement:

$$[q^n] \left(T_{1/2,k}(q) - T_{r,k}(q) \right) \geq 0 \quad (r \in \mathbb{Q}_{>0}, k \geq 1, n \geq 0). \quad (8.1)$$

This completes the proposed proof of the full Han–Xiong conjecture. $\square$

Appendix. AI use and supplementary verification

Author and AI use. Justin Du directed the AI-assisted investigation and preparation of this preprint. Generative AI contributed substantially to the proposed mathematical argument, manuscript text, and verification programs. OpenAI Codex was used for subsequent manuscript revision, internal review, and preparation of this release. The author is responsible for this preprint and is the contact for questions and corrections.

Review status. The argument has undergone internal AI-assisted mathematical review and exact-arithmetic checks. It has not undergone external human peer review or proof-assistant formalization. The word candidate records this status; the proof does not intentionally assume

an unresolved lemma. Independently implemented checking programs provide arithmetic cross-checks, rather than independent human peer review.

Verification materials. All inequalities needed for the main proof are displayed in Sections 1–7. The variable-kernel arguments use exact coefficient formulas and polynomial inequalities, without a finite parameter cutoff or symbolic case enumeration as a premise. The accompanying source package contains the TeX manuscript, an analytic arithmetic checker, four supplementary certificate programs, and two exact certificate files. The analytic checker verifies the displayed polynomial identities and the fixed kernel’s initial coefficients. The other programs provide alternative checks of the two variable kernels, summarized below.

Supplementary check	Exact result
Old K_j certificate, $j \geq 2$	3,468 symbolic cases; 29,424 nonzero coefficients; minimum 72 (scaling by 6)
Uniform Q_a certificate, $a \geq 3$	1,884 symbolic cases; 15,312 nonzero coefficients; minimum 66 (scaling by 6)
Independent Q_a BigInt verifier	Same 1,884 cases and 15,312 coefficients; scaling by 288 gives minimum 3,168
Complete formal proof in a proof assistant	Not supplied

The software checks do not replace the coefficientwise-order arguments in the text. Each order-preserving multiplication is justified by an explicitly nonnegative series or a stated identity. The parameter ranges previously established in [1]–[3] are not assumptions of the proposed proof. The mathematical argument is intended to cover all parameters in Theorem 1.1; the verification programs are supplementary materials.

References

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